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Calculus Rules for Global Approximate Minima and Applications to Approximate Subdifferential Calculus

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Abstract. We provide calculus rules for global approximate minima concerning usual operations on functions. The formulas we obtain are then applied to approximate subdifferential calculus. In this way, new results are presented, for example on the approximate subdifferential of a deconvolution, or on the subdifferential of an upper envelope of convex functions.

Key words: Convex and nonconvex duality, asymptotical analysis, approximate subdifferential, approximate minimizer.

0. Introduction

The concept of approximate minimum is very important in optimization for theorical as well as practical reasons. For example, due to some imprecision on the data, the question of finding optimal solutions of the problem

minimize a real function f over the set X

may be considered as solved if we are able to produce elements x of X such that

$$x \in \varepsilon\text{-argmin } f := \{x \in X : f(x) \leq \inf_X f + \varepsilon\}$$

for $\varepsilon > 0$ sufficiently small. On the other hand, the advantage in dealing with approximate minima of a minorized function f, is the nonvoidness of ε-argmin f for any $\varepsilon > 0$.

In the first part of the paper we give some elementary calculus rules for approximate minima. The results we obtain are shown to be applicable to subdifferential calculus in the second part. This is not too surprising as approximate minima and approximate subdifferentials are linked together by Legendre–Fenchel transform. However, the notion of approximate minima is intrinsically more general than the one concerning the approximate subdifferential. The reason for this is very simple: defining approximate minima on X does not require any structure on the set X. Let us compare in details these two concepts when X is a locally convex topological space (ℓ.c.s.) paired in separated duality with another ℓ.c.s. W; in such a frame, which is the more natural for dealing with Legendre–Fenchel duality theory, the conjugate of an extended real valued function $f : X \to \overline{\mathcal{R}}$ is given by

$$f^*(w) = \sup_{x \in X}\{< x, w > -f(x)\} \quad \text{for any} \quad w \in W .$$

The conjugate of an extended real valued function on W is defined in analogous terms. The biconjugate of f is then $f^{**} = (f^*)^*$. It is well known that, for a proper function $f : X \to \overline{\mathcal{R}}$ (proper means f is not identically $+\infty$ and never takes the value $-\infty$), $f = f^{**}$ iff f belongs to the set $\Gamma_0(X)$ of lower-semicontinuous (ℓ.s.c.) proper convex functions on X. Given $\varepsilon \geq 0$, and a function $f : X \to \overline{\mathcal{R}}$, finite at x, one defines the ε-subdifferential of f at x as follows:

$$\partial_\varepsilon f(x) = \{w \in W : \forall u \in X : f(u) - f(x) \geq < u - x, w > -\varepsilon\} .$$

Without further assumptions, one has,

$$\partial_\varepsilon f(x) = \{w \in W : f^*(w) - < x, w > +f(x) \leq \varepsilon\} . \tag{1}$$

Of course, a similar notion does exist for functions defined on W.

On the other hand, the ε-argmin of an extended real valued function φ on a set Z is defined, when $m := \inf_{z \in Z} \varphi(z) \in \mathcal{R}$, by

$$\varepsilon\text{-argmin } \varphi = \{z \in Z : \varphi(z) \leq m + \varepsilon\} .$$

If $m = -\infty$ we set ε-argmin $\varphi = \emptyset$ for all $\varepsilon \geq 0$, and also r-argmin $\varphi = \emptyset$ (resp. $\partial_r f(x) = \emptyset$) for all $r < 0$.

For any $x \in f^{-1}(\mathcal{R}), w \in W$, the obvious relation

$$x \in \varepsilon\text{-argmin}(f - <, w >) \Longleftrightarrow w \in \partial_\varepsilon f(x)$$

gives a first connexion between the concepts of approximate minima and approximate subdifferential. When $f - <, w >$ is minorized we also have

$$\varepsilon\text{-argmin}(f - <, w >) = \{u \in X : f(u) - < u, w > +f^*(w) \leq \varepsilon\} , \tag{2}$$

a formula that may be compared with (1).

Notice that the implication,

$$x \in \varepsilon\text{-argmin } f \Longrightarrow x \in \partial_\varepsilon\, f^*(0) ,$$

is always true, but the converse,

$$x \in \partial_\varepsilon\, f^*(0) \Longrightarrow x \in \varepsilon\text{-argmin } f ,$$

is true if and only if $f(x) = f^{**}(x)$; this condition does not systematically hold in the application we have in view. However:

PROPOSITION 0.1. *Let $f : X \to \overline{\mathbb{R}}$ and $x \in X$ be such that*

$$-\infty < f^{**}(x) \leq f(x) < +\infty .$$

Then, setting $\delta(x) = f(x) - f^{**}(x)$, we have, for all $\varepsilon \geq 0$,

$$\begin{aligned} \partial_\varepsilon f(x) &= \emptyset \quad \text{for} \quad \varepsilon \in [0, \delta(x)[\\ \partial_\varepsilon f(x) &= \partial_{\varepsilon-\delta(x)} f^{**}(x) \quad \text{for} \quad \varepsilon \geq \delta(x) . \end{aligned}$$

In particular,

$$x \in \varepsilon\text{-argmin}\, f \Longleftrightarrow x \in \partial_{\varepsilon-\delta(x)} f^*(0) .$$

Proof. For all $w \in W$, $w \in \partial_\varepsilon f(x)$ iff

$$f(u) \geq f(x) + < u - x, w > -\varepsilon, \text{ for any } \quad u \in X .$$

As f and f^{**} have the same affine minorants, the line above is equivalent to

$$f^{**}(u) \geq f(x) + < u - x, w > -\varepsilon = f^{**}(x) + < u - x, w > -\varepsilon + \delta(x),$$

$$\text{for any} \quad u \in X ,$$

that is to say

$$w \in \partial_{\varepsilon-\delta(x)} f^{**}(x) .$$

For the last equivalence we have, as $f^* = f^{***}$,

$$0 \in \partial_\varepsilon f(x) = \partial_{\varepsilon-\delta(x)} f^{**}(x) \Longleftrightarrow x \in \partial_{\varepsilon-\delta(x)} f^*(0) .$$ ∎

The next proposition shows explicitly that the problem of computing an approximate subdifferential is reducible to an approximate argmin problem.

PROPOSITION 0.2. *Let f, x, and $\delta(x)$ as in Proposition 1. Then, for any $\varepsilon \geq 0$,*

$$\partial_\varepsilon f(x) = (\varepsilon - \delta(x))\text{-argmin}(f^* - < x, . >) .$$

Proof. For each $w \in W$ one has $w \in \partial_\varepsilon f(x) = \partial_{\varepsilon-\delta(x)} f^{**}(x)$ iff $x \in \partial_{\varepsilon-\delta(x)} f^*(w)$, iff $0 \in \partial_{\varepsilon-\delta(x)}(f^* - < x, . >)(w)$, iff $w \in (\varepsilon - \delta(x))\text{-argmin}(f^* - < x, . >)$. ∎

1. Calculus Rules for Approximate Minima

In this section we give formulas concerning the approximate minima of usual operations on functions like sum, difference, composition, inf-convolution, deconvolution, upper envelope, lower envelope... To this end we shall use some notations and properties.

The addition (resp. subtraction) of extended real numbers will be taken in the following sense:

$$\forall a, b \in \overline{\mathcal{R}} : a + b = +\infty \quad \text{whenever} \quad a = +\infty \quad \text{or} \quad b = +\infty$$

$$(\text{resp. } a - b = a + (-b)) \, .$$

When dealing with inequalities, we systematically use the following properties ([18] p.118), valid for all $a, b, c \in \overline{\mathcal{R}}$:

$$a + b \leq c \iff \exists d, e \in \overline{\mathcal{R}} : \ a \leq d, \quad b \leq e, \quad d + e = c \tag{3}$$

$$a + b < c \iff \exists d, e \in \overline{\mathcal{R}} : \ a < d, \quad b < e, \quad d + e = c \, . \tag{4}$$

If, moreover, a and b are nonnegative, then d, e above can be chosen nonnegative.

The domain of an extended real valued function $\varphi : Z \to \overline{\mathcal{R}}$ is denoted by,

$$\text{dom}\, \varphi = \{z \in Z : \varphi(z) < +\infty\} \, ,$$

while,

$$E(\varphi) = \{(z, r) \in Z \times \mathcal{R} : \varphi(z) \leq r\} \, ,$$

will be the epigraph of φ.

1.1. A DIRECT APPROACH

We begin with approximate minima of a sum of two minorized functions.

PROPOSITION 1.1. *Assume that the functions,*

$$\varphi, \psi : Z \to \mathcal{R} \cup \{+\infty\} \, ,$$

are minorized on the set Z, and that $\varphi + \psi$ is proper. Then, for any $\varepsilon \geq 0$,

$$\varepsilon\text{-argmin}\, (\varphi + \psi) = \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \gamma - \alpha - \beta}} \varepsilon_1\text{-argmin}\, \varphi \quad \cap \quad \varepsilon_2\text{-argmin}\, \psi \, ,$$

with $\alpha = \inf_Z \ \varphi, \quad \beta = \inf_Z \ \psi, \quad \gamma = \inf_Z (\varphi + \psi)$.

Proof. Observe that $\gamma \geq \alpha + \beta$. For any $z \in Z$, $z \in \varepsilon$-argmin$(\varphi + \psi)$ means

$$\varphi(z) - \alpha + \psi(z) - \beta \leq \varepsilon + \gamma - \alpha - \beta \, .$$

As $\varphi(z) - \alpha$ and $\psi(z) - \beta$ are nonnegative, (3) shows that the line above is equivalent to the existence of $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon + \gamma - \alpha - \beta$, and $\varphi(z) - \alpha \leq \varepsilon_1, \psi(z) - \beta \leq \varepsilon_2$, that is to say

$$z \in \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \gamma - \alpha - \beta}} \varepsilon_1\text{-argmin } \varphi \cap \varepsilon_2\text{-argmin } \psi . \qquad \blacksquare$$

The computation of the approximate argmin of a difference of functions involves the difference of sets: for A, B subsets of Z we denote

$$A \setminus B = \{z \in Z : z \in A \quad \text{and} \quad z \notin B\} .$$

PROPOSITION 1.2. *Let $\varphi, \psi : Z \to \mathcal{R} \cup \{+\infty\}$ be proper. Assume that ψ and $\varphi - \psi$ are minorized on Z. Then, for any $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin } (\varphi - \psi) = \bigcap_{\eta > \varepsilon} \bigcup_{r > 0} r\text{-argmin } \varphi \setminus (r + \alpha - \beta - \delta - \eta)\text{-argmin } \psi ,$$

with $\alpha = \inf_Z \varphi$, $\beta = \inf_Z \psi$, $\delta = \inf_Z \varphi - \psi$.

Proof. Let us note that φ is necessarily minorized with $\alpha \geq \delta + \beta$. Let z be any ε-minimizer of $\varphi - \psi$. For all $\eta > \varepsilon$ one has $\varphi(z) - \psi(z) < \eta + \delta$, so that $(\varphi(z) - \alpha) + (\beta - \psi(z)) < \eta + \delta + \beta - \alpha.$ By (4) there exist $r, s \in \mathcal{R}$ such that $\varphi(z) - \alpha < r$, $\beta - \psi(z) < s$, $r + s = \eta + \delta + \beta - \alpha.$ Now, r is necessarily positive and we have

$$z \in \bigcap_{\eta > \varepsilon} \bigcup_{r > 0} r\text{-argmin } \varphi \setminus (r + \alpha - \beta - \delta - \eta)\text{-argmin } \psi := S .$$

Conversely, let us take $z \in S$. For any $\eta > \varepsilon$ there exists $r > 0$ such that $\varphi(z) \leq \alpha + r$ and $\psi(z) > r + \alpha - \delta - \eta$, so that $\varphi(z) - \psi(z) < \delta + \eta$. Hence $\varphi(z) - \psi(z) \leq \delta + \varepsilon$, that is, $z \in \varepsilon\text{-argmin}\varphi - \psi$. $\blacksquare$

Propositions 3, 4, and 5 below concern two important classes of functions: composite functions and marginal functions with explicit or implicit constraints. Here X is another set, $G : X \to Z$ an application, and Γ: $X \rightrightarrows Z$ a multiapplication; to each function $\varphi : Z \to \mathcal{R} \cup \{+\infty\}$ are associated the composite function,

$$\varphi \circ G : X \to \mathcal{R} \cup \{+\infty\}, \quad (\varphi \circ G)(x) = \varphi(G(x)) ,$$

and the marginal function,

$$\varphi_\Gamma : X \to \mathcal{R} \cup \{+\infty\} \quad \varphi_\Gamma(x) = \inf\{\varphi(z) : z \in \Gamma x\} ,$$

with the usual convention inf $\emptyset = +\infty$. We note G^{-1} (resp. Γ^{-1}) the inverse relation of G (resp. Γ).

PROPOSITION 1.3. *Assume that φ is minorized and dom $\varphi \cap G(X) \neq \emptyset$. Then, for any $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin}\,\varphi \circ G = G^{-1}((\varepsilon + \beta - \alpha)\text{-argmin}\,\varphi)\ ,$$

with $\alpha = \inf_Z \varphi$, $\beta = \inf_X \varphi \circ G$.

Proof. First we note that $-\infty < \alpha \leq \beta < +\infty$. Then, x is an ε-minimizer of $\varphi \circ G$ iff $\varphi(G(x)) \leq \beta + \varepsilon$, or $\varphi(G(x)) \leq \alpha + \beta - \alpha + \varepsilon$, or $G(x) \in (\varepsilon + \beta - \alpha)\text{-argmin}\,\varphi$, that is $x \in G^{-1}((\varepsilon + \beta - \alpha)\text{-argmin}\,\varphi)$. ∎

PROPOSITION 1.4. *Assume that φ is minorized and dom $\varphi \cap \Gamma(X) \neq \emptyset$. Then, for any $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin}\,\varphi_\Gamma = \bigcap_{\eta > \varepsilon} \Gamma^{-1}((\gamma - \alpha + \eta)\text{-argmin}\,\varphi)\ ,$$

with $\alpha = \inf_Z \varphi$, $\gamma = \inf_{\Gamma(X)} \varphi$.

Proof. Here we have $-\infty < \alpha \leq \gamma < +\infty$. Moreover, x is an ε-minimizer of φ_Γ iff for each $\eta > 0$ there exists $z \in \Gamma(x)$ such that $\varphi(z) \leq \gamma + \varepsilon + \eta = \alpha + \gamma - \alpha + \varepsilon + \eta$, iff $x \in \bigcap_{\eta > 0} \Gamma^{-1}((\gamma - \alpha + \varepsilon + \eta)\text{-argmin}\,\varphi)$. ∎

When no explicit constraints occurs, that is when,

$$f(x) = \inf_{z \in Z} F(x, z)\ ,$$

with $F : X \times Z \to \mathcal{R} \cup \{+\infty\}$, Proposition 4 can be applied by putting F instead of φ and Γ: $X \rightrightarrows Z, \Gamma(x) = \{x\} \times Z$. We then obtain, denoting by P the projection of $X \times Z$ onto X:

COROLLARY 1.5. *Assume that F is proper and minorized. Then for any $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin}\, f = \bigcap_{\eta > \varepsilon} P(\eta\text{-argmin}\, F)\ .$$

Proof. Here, $\inf_X f = \inf_{X \times Z} F$ and $P = \Gamma^{-1}$. ∎

Another formula is needed when we are faced with the following situation: $(\varphi_i)_{i \in I}$ is a family of proper functions $\varphi_i : Z \to \mathcal{R} \cup \{+\infty\}$ indexed by I, an arbitrary nonvoid set. We will express the set of ε-minimizers of $\varphi = \inf_{i \in I} \varphi_i$ in terms of approximate minimizers of φ_i. Let us set

$$\alpha_i = \inf_Z \varphi_i \quad \text{for any} \quad i \in I, \quad \alpha = \inf_Z \varphi\ .$$

Then $\alpha = \inf_{i \in I} \alpha_i$. If φ is minorized, that is if the φ_i are equi-minorized, the set

$$I(\eta) = \{i \in I : \alpha_i \leq \alpha + \eta\}$$

is nonvoid for any $\eta > 0$.

PROPOSITION 1.6. *Let* $(\varphi_i)_{i\in I}$, $\varphi = \inf_{i\in I}\varphi_i$, $(\alpha_i)_{i\in I}, \alpha$, *and* $I(\eta)$ *be as above. Then, for any* $\varepsilon \geq 0$,

$$\varepsilon\text{-argmin}\,\varphi = \bigcap_{\eta>\varepsilon} \bigcup_{i\in I(\eta)} (\alpha - \alpha_i + \eta)\text{-argmin}\,\varphi_i\ .$$

Proof. For each $z \in Z$, the following lines are equivalent:

$$z \in \varepsilon\text{-argmin}\,\varphi$$

$$\forall\eta > \varepsilon \quad \exists i \in I : \varphi_i(z) \leq \alpha + \eta$$

$$\forall\eta > \varepsilon \quad \exists i \in I : z \in (\alpha - \alpha_i + \eta)\text{-argmin}\,\varphi_i$$

$$\forall\eta > \varepsilon \quad \exists i \in I(\eta) : z \in (\alpha - \alpha_i + \eta)\text{-argmin}\,\varphi_i$$

$$z \in \bigcap_{\eta>\varepsilon} \bigcup_{i\in I(\eta)} (\alpha - \alpha_i + \eta)\text{-argmin}\,\varphi_i\ .$$

■

The presence of α_i in formula above may be avoided, at least for $\varepsilon = 0$. This fact is a mere consequence of

LEMMA 1.7. *For all* $\varepsilon \geq 0$*:*

$$\bigcap_{\eta>\frac{\varepsilon}{2}} \bigcup_{i\in I(\eta)} \eta\text{-argmin}\,\varphi_i \subset \varepsilon\text{-argmin}\,\varphi \subset \bigcap_{\eta>\varepsilon} \bigcup_{i\in I(\eta)} \eta\text{-argmin}\,\varphi_i\ .$$

Proof. Assume that $z \notin \varepsilon$-argmin φ; then there exists $\eta > \varepsilon/2$ such that $\varphi(z) > \alpha + 2\eta$. Now, for any $i \in I(\eta)$, $\varphi_i(z) \geq \varphi(z) > \alpha_i + \eta$, so that $z \notin \eta$-argminφ_i, and the first inclusion is proved. To prove the second inclusion let us consider any $\eta > \varepsilon$. Then, for all $z \in \varepsilon$-argmin φ, we have $\varphi(z) < \alpha + \eta$ and there exists $i \in I$ such that $\alpha_i \leq \varphi_i(z) < \alpha + \eta = \alpha_i + (\alpha - \alpha_i) + \eta \leq \alpha_i + \eta$, so that $z \in \eta$-argmin φ_i, with $i \in I(\eta)$. ■

In the particular case $\varepsilon = 0$ we get:

PROPOSITION 1.8. *Let* $(\varphi_i)_{i\in I}$ *be a family of equi-minorized proper functions on the set* Z, $\varphi = \inf_{i\in I}\varphi_i$. *Then,*

$$\textit{argmin}\,\varphi = \bigcap_{\eta>0} \bigcup_{i\in I(\eta)} \eta\text{-argmin}\,\varphi_i\ .$$

In the case when,

$$\forall z \in \text{ argmin}\,\varphi\ , \quad \exists i \in I : \varphi(z) = \varphi_i(z)\ ,$$

we have,

$$\operatorname{argmin} \varphi = \bigcup_{i \in I(0)} \operatorname{argmin} \varphi_i \ ,$$

where, for any $\eta \geq 0$ *,*

$$I(\eta) = \{i \in I : \inf_Z \varphi_i \leq \inf_Z \varphi + \eta\} \ .$$

Proof. We only have to prove the second formula for which the inclusion $\supset$ is clear. Let us take $z \in$ argmin φ. By assumption, there exists $i \in I$ such that,

$$\inf_Z \varphi_i \leq \varphi_i(z) = \varphi(z) = \inf_Z \varphi \leq \inf_Z \varphi_i \ ,$$

hence, $i \in I(0)$ and $z \in$ argmin φ_i . ■

We now consider the case of an upper envelope $\psi = \sup_{i \in I} \varphi_i$ where $\varphi_i : Z \to \mathcal{R} \cup \{+\infty\}$ is proper and minorized for any $i \in I$:

$$\forall i \in I : \inf_Z \varphi_i = \alpha_i \in \mathcal{R} \ .$$

We assume that ψ is proper; hence,

$$+\infty > \alpha := \inf_Z \psi \geq \sup_{i \in I} \alpha_i := \beta > -\infty \ .$$

Let us take $\varepsilon \geq 0$ and $z \in \varepsilon$- argmin ψ. We then have,

$$\varphi_i(z) \leq \alpha_i + \varepsilon + \alpha - \alpha_i \qquad \text{for any} \quad i \in I \ ,$$

so that,

$$z \in \bigcap_{i \in I} (\varepsilon + \alpha - \alpha_i)\text{-argmin}\, \varphi_i \ .$$

By introducing for each $\eta > 0$ the nonvoid set,

$$I_\eta = \{i \in I : \alpha_i \geq \beta - \eta\} \ ,$$

we easily see that,

$$z \in \bigcap_{\eta > 0} \ \bigcap_{i \in I_\eta} (\varepsilon + \alpha - \beta + \eta)\text{-argmin}\, \varphi_i \ ,$$

so that,

$$\varepsilon\text{- argmin}\, \psi \subset \bigcap_{\eta > 0} \ \bigcap_{i \in I_\eta} (\varepsilon + \alpha - \beta + \eta)\text{-argmin}\, \varphi_i \ .$$

In fact the converse inclusion also holds. To see this assume that $z \notin \varepsilon$-argmin ψ. Then there exist $i \in I$ and $\delta > 0$ such that $\varphi_i(z) > \alpha + \varepsilon + \delta$. By taking $\eta = \beta - \alpha_i + \delta$ we have $\eta > 0, i \in I_\eta$, and $\varphi_i(z) > \alpha_i + \varepsilon + \alpha - \beta + \eta$, so that $z \notin (\varepsilon + \alpha - \beta + \eta)$-argmin φ_i .

Hence we have proved:

PROPOSITION 1.9. *Let* $(\varphi_i)_{i \in I}$, $\psi = \sup_{i \in I} \varphi_i$, α, β, *and* I_η *be as above. Then, for any* $\varepsilon \geq 0$,

$$\varepsilon\text{-argmin}\, \psi = \bigcap_{\eta > 0} \bigcap_{i \in I_\eta} (\varepsilon + \alpha - \beta + \eta)\text{-argmin}\, \varphi_i .$$

Up to now, Z was an arbitrary set; let us supposed that Z is endowed with a structure of additive group.

For any proper functions $f, g : Z \to \mathcal{R} \cup \{+\infty\}$ the inf-convolution ([18], [21]) of f and g is given by

$$(f \square g)(z) = \inf_{u \in Z} f(z - u) + g(u) \quad \text{for any} \quad z \in Z .$$

A kind of inverse operation to the preceding one has been introduced ([11], [17], ...) for solving inf-convolutive equations:

$$\text{find} \quad \xi \in \overline{\mathcal{R}}^Z \qquad \text{such that} \quad g \square \xi = f .$$

It has been shown that such an equation admits solution iff the deconvolution of f and g, namely

$$(f \boxminus g)(z) = \sup\{f(z + u) - g(u) : u \in \operatorname{dom} g\}, \quad \forall z \in Z ,$$

is one of them. Geometrically, the deconvolution is linked with the star difference of sets. Recall that, given two subsets C, D of Z, the star difference of C and D is defined as follows,

$$C \overset{*}{-} D = \bigcap_{x \in D} C - x = \{z \in Z : z + D \subset C\} .$$

It is known ([26] Prop. 6) that the epigraph of $f \boxminus g$ coincides with the star difference of the epigraphs of f and g:

$$E(f \boxminus g) = E(f) \overset{*}{-} E(g) .$$

Also,

$$\operatorname{dom} f \boxminus g \subset \operatorname{dom} f \overset{*}{-} \operatorname{dom} g .$$

Then, one may imagine that the approximate minima of $f \boxminus g$ can also be expressed in terms of the star difference of the approximate minima of f and g. The proposition below makes this idea more precise.

PROPOSITION 1.10. *Assume $f, g : Z \to \mathcal{R} \cup \{+\infty\}$ proper, minorized. Then for any $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin}\, f \boxminus g = \bigcap_{\eta>0} (\varepsilon + \alpha - \beta + \eta)\text{-argmin}\, f \overset{*}{-} \eta\text{-argmin}\, g\ ,$$

with $\alpha = \inf_Z f \boxminus g \geq \inf_Z f - \inf_Z g = \beta$.

Proof. We apply Proposition 9 by putting

$$I = \operatorname{dom} g, \quad \varphi_u(z) = f(z+u) - g(u) \quad \text{for any} \quad u \in \operatorname{dom} g, \quad z \in Z\ .$$

Then $\psi = f \boxminus g$, $I_\eta = \eta$-argmin g, λ-argmin $\varphi_u = \lambda$-argmin $f - u$ for any $\lambda \geq 0$, and we have:

$$\varepsilon\text{-argmin}\, f \boxminus g = \bigcap_{\eta>0} \bigcap_{u \in \eta\text{-argmin}\, g} [(\varepsilon + \alpha - \beta + \eta)\text{-argmin}\, f] - u$$

$$= \bigcap_{\eta>0} (\varepsilon + \alpha - \beta + \eta)\text{-argmin}\, f \overset{*}{-} \eta\text{-argmin}\, g\ . \qquad \blacksquare$$

Of course, Propositions 6 and 8 can be used to compute the approximate minima of an inf-convolution (or epigraphical sum). Such a question has been considered in [3] [1], where some estimations are given. Here we give exact general formulas.

PROPOSITION 1.11. *Assume $f, g : Z \to \mathcal{R} \cup \{+\infty\}$ proper, minorized. Then,*

$$\text{argmin}\, f \Box g = \bigcap_{\eta>0} \eta\text{-argmin}\, f + \eta\text{-argmin}\, g\ ,$$

where $+$ denotes the algebraic sum of sets.

Proof. Apply Proposition 8 by taking $I = \operatorname{dom} g$, $\varphi_u(z) = f(z-u) + g(u)$ for any $(u, z) \in \operatorname{dom} g \times Z$. Then $\varphi = f \Box g$, $\inf_Z \varphi = \inf_Z f + \inf_Z g$, $I(\eta) = \eta$-argmin g, η-argmin $\varphi_u = \eta$-argmin $f + u$, and we have:

$$\text{argmin}\, f \Box g = \bigcap_{\eta>0} \bigcup_{u \in \eta\text{-argmin}\, g} \eta\text{-argmin}\, f + u$$

$$= \bigcap_{\eta>0} \eta\text{-argmin}\, f + \eta\text{-argmin}\, g\ . \qquad \blacksquare$$

Due to the special form of the inf-convolution, we have, for $\varepsilon > 0$, an improvement of Proposition 6:

PROPOSITION 1.12. *Assume $f, g : Z \to \mathcal{R} \cup \{+\infty\}$ proper, minorized. Then, for any $\varepsilon > 0$,*

$$\varepsilon\text{-argmin } f \,\Box\, g = \bigcap_{\delta > \varepsilon} \bigcup_{\substack{\delta_1 \geq 0\, \delta_2 \geq 0 \\ \delta_1 + \delta_2 = \delta}} \delta_1\text{-argmin } f + \delta_2\text{-argmin } g \ .$$

Proof. Recall that $\inf_Z f \,\Box\, g = \inf_Z f + \inf_Z g$. Now let z be a ε-minimizer of $f \,\Box\, g$, and $\delta > \varepsilon$. There exists $u \in Z$ such that $f(z-u) + g(u) < \inf_Z f + \inf_Z g + \delta$, or $f(z-u) - \inf_Z f + g(u) - \inf_Z g < \delta$. By (4) there exist $\delta_1 \geq 0$, $\delta_2 \geq 0$ such that $\delta_1 + \delta_2 = \delta$, $0 \leq f(z-u) - \inf_Z f \leq \delta_1$, $0 \leq g(u) - \inf_Z g \leq \delta_2$ so that

$$z = (z-u) + u \ \in \ \delta_1\text{-argmin } f + \delta_2\text{-argmin } g \ ,$$

$$z \in \bigcap_{\delta > \varepsilon} \bigcup_{\substack{\delta_1 \geq 0,\, \delta_2 \geq 0 \\ \delta_1 + \delta_2 = \delta}} \delta_1\text{-argmin } f + \delta_2\text{-argmin } g \ .$$

Conversely let z be as above; then for any $\delta > \varepsilon$, there exist $\delta_1 \geq 0$, $\delta_2 \geq 0$, $\delta_1 + \delta_2 = \delta$, $z_1 \in \delta_1$-argmin f, $z_2 \in \delta_2$-argmin g such that $z = z_1 + z_2$; hence

$$(f \,\Box\, g)(z) \leq f(z_1) + g(z_2) \leq \delta_1 + \delta_2 + \inf_Z f + \inf_Z g = \delta + \inf_Z f \,\Box\, g \ ,$$

that is $z \in \delta$-argmin $f \,\Box\, g$ for any $\delta > \varepsilon$, so that $z \in \varepsilon$-argmin $f \,\Box\, g$. ∎

In the case when the inf-convolution $f \,\Box\, g$ is exact, that is for any $z \in Z$ there exists $u \in Z$ such that,

$$(f \,\Box\, g)(z) = f(z-u) + g(u) \ ,$$

the previous formulas can be simplified as follows (see also [2]).

PROPOSITION 1.12 bis. *Let $f, g : Z \to \mathcal{R} \cup \{+\infty\}$ be proper and minorized. Assume that $f \,\Box\, g$ is exact. Then, for any $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin } f \,\Box\, g = \bigcup_{\substack{\varepsilon_1 \geq 0,\, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \varepsilon_1\text{-argmin } f + \varepsilon_2\text{-argmin } g \ .$$

In particular,

$$\text{argmin } f \,\Box\, g = \text{argmin } f \ + \ \text{argmin } g \ .$$

Proof. As $f \,\Box\, g$ is exact, $z \in Z$ is a ε-minimizer of $f \,\Box\, g$ iff there exist $u, v \in Z$ such that $u + v = z$ and $f(u) + g(v) \leq \inf_Z (f \,\Box\, g) + \varepsilon$ or, equivalently,

$(f(u)-\inf_Z f)+(g(v)-\inf_Z g) \leq \varepsilon$. By (3) this is equivalent to the existence of $\varepsilon_1 \geq 0, \varepsilon_2 \geq 0$ such that $\varepsilon_1+\varepsilon_2 = \varepsilon$ and $(f(u)-\inf_Z f) \leq \varepsilon_1, (g(v)-\inf_Z g) \leq \varepsilon_2$. In other words:

$$z \in \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \varepsilon_1\text{-argmin } f + \varepsilon_2\text{-argmin } g \ . \qquad \blacksquare$$

We now give a result of topological nature.

PROPOSITION 1.13. *Let Z be a topological space, $\varphi : Z \to \mathcal{R} \cup \{+\infty\}$ proper, minorized, $\overline{\varphi}$ the lower semicontinuous regularized of φ. Then, for any $\varepsilon > 0$,*

$$\varepsilon\text{-argmin}\, \overline{\varphi} = \bigcap_{\delta > \varepsilon} c\ell(\delta\text{-argmin}\, \varphi) \ .$$

Proof. Let us set $m = \inf_Z \varphi$. As $\overline{\varphi}$ is the greatest ℓ.s.c. minorant of φ, we have $m \leq \overline{\varphi} \leq \varphi$, hence $\inf_Z \overline{\varphi} = m$. We then obtain,

$$\varepsilon\text{-argmin}\, \overline{\varphi} = \{z \in Z : \overline{\varphi}(z) \leq m + \varepsilon\} = \bigcap_{\delta > \varepsilon} c\ell\{z \in Z : \varphi(z) \leq m + \delta\}$$

$$= \bigcap_{\delta > \varepsilon} c\ell(\delta\text{-argmin}\, \varphi) \ . \qquad \blacksquare$$

We end this section with a result concerning the minimizers of the ℓ.s.c. convex hull of a given function. Let $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ be a ℓ.s.c. proper function that we suppose coercive in the following sense,

$$\exists \varepsilon > 0, \quad \exists c \in \mathcal{R} : f \geq \varepsilon \| \ \| - c \ . \tag{5}$$

This assumption ensures that the asymptotic function f_∞ of f, that is

$$f_\infty(u) = \liminf_{(t,v)\to(0_+,u)} tf\left(\frac{v}{t}\right) \quad \text{for any} \quad u \in \mathcal{R}^n \ ,$$

satisfies

$$\text{argmin}\, f_\infty = \{0\} \ . \tag{6}$$

Indeed, (5) implies $f_\infty \geq \varepsilon \| \ \|$, and, as $f_\infty(0) = 0$, we easily get (6).

Now, let us consider the biconjugate (or closed convex hull) f^{**} of f; the function f being ℓ.s.c. and epi-pointed in the sense of [4], we have by ([4] p. 18),

$$\text{argmin}\, f^{**} = \text{co argmin}\, f + \text{co argmin}\, f_\infty \tag{7}$$

and by (6),

$$\text{argmin} f^{**} = \text{co argmin}\, f \ .$$

Hence we have proved:

PROPOSITION 1.14. *Let $f : \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ be a proper ℓ.s.c. function satisfying* (5). *Then,*

$$\operatorname{argmin} f^{**} = \operatorname{co} \operatorname{argmin} f \ .$$

1.2. THE USE OF A DUAL VARIABLE

We return to the problem of finding the approximate minimizers of a sum, yet considered in Proposition 1. In this proposition φ and ψ were defined on an abstract set. Here we suppose that they are defined on a ℓ.c.s. Z paired in duality with another ℓ.c.s. Y, but we don't retain the fact that φ and ψ are minorized: we just assume that $\varphi + \psi$ is proper and minorized, and set

$$m = \inf_Z (\varphi + \psi) \ .$$

To each $y \in \operatorname{dom} \varphi^* \cap (-\operatorname{dom} \psi^*) := \Delta$, we associate the nonnegative real number,

$$\delta(y) = \varphi^*(y) + \psi^*(-y) + m \ .$$

Let us consider $y \in \Delta$. An element z of Z is an ε-minimizer of $\varphi + \psi$ iff

$$[\varphi(z) - < z, y > + \varphi^*(y)] + [\psi(z) + < z, y > + \psi^*(-y)] \leq \delta(y) + \varepsilon \ .$$

Each function of z into the brackets admits zero for infimum, while the infimum of their sum is $\delta(y)$. Applying Proposition 1 and (2) we get:

PROPOSITION 1.15. *For any functions $\varphi, \psi : Z \to \mathcal{R} \cup \{+\infty\}$ such that $\varphi + \psi$ is proper and minorized, we have for all $\varepsilon \geq 0$ and all $y \in$ $\operatorname{dom} \varphi^* \cap (-\operatorname{dom} \varphi^*)$,*

$$\varepsilon\text{-argmin}(\varphi+\psi) = \bigcup_{\substack{\varepsilon_1 \geq 0,\, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \delta(y)}} \varepsilon_1\text{-argmin}(\varphi - <, y>) \cap \varepsilon_2\text{-argmin}(\psi + <, y>),$$

where $\delta(y) = \varphi^*(y) + \psi^*(-y) + \inf_Z(\varphi + \psi)$.

A classical way to obtain the existence of $\overline{y} \in \Delta$ such that $\delta(\overline{y}) = 0$ is to require the convexity of φ and ψ together with the continuity of φ at a point of dom ψ ([20]). In such a case:

COROLLARY 1.16. *Assume that φ and ψ are convex with φ finite and continuous at a point of dom ψ. Then there exists $\overline{y} \in \operatorname{dom} \varphi^* \cap (-\operatorname{dom} \psi^*)$ such that, for all $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin}(\varphi + \psi) = \bigcup_{\substack{\varepsilon_1 \geq 0,\, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \varepsilon_1\text{-argmin}(\varphi - <, \overline{y}>) \cap \varepsilon_2\text{-argmin}(\psi + <, \overline{y}>).$$

In particular, for such a $\overline{y}$,

$$\operatorname{argmin}(\varphi+\psi) = \operatorname{argmin}(\varphi- <,\overline{y}>) \cap \operatorname{argmin}(\psi+ <,\overline{y}>) .$$

A similar approach may be used for the approximate minimizers of a difference functions $\varphi-\psi$, with $\varphi-\psi$ proper and minorized. Following Toland–Singer duality scheme ([23] [24]), we associate to each $y \in \operatorname{dom} \varphi^* \cap \operatorname{dom} \psi^*$ the nonnegative real number,

$$\delta(y) = \psi^*(y) - \varphi^*(y) - \inf_Z(\varphi - \psi) .$$

An element z of Z is a ε-minimizer of $\varphi - \psi$ iff,

$$[\varphi(z)- < z,y > +\varphi^*(y)] - [\psi(z)- < z,y > +\psi^*(y)] \leq \varepsilon - \delta(y) .$$

Taking into account the fact that both functions of z into the brackets have zero for infimum, while their difference admits $-\delta(y)$ for infimum, Proposition 1.2 and (2) give us:

PROPOSITION 1.17. *Let $\varphi,\psi : Z \to \mathcal{R} \cup \{+\infty\}$ be proper functions such that $\varphi-\psi$ is proper and minorized. For any $y \in \operatorname{dom} \varphi^* \cap \operatorname{dom} \psi^*$ and any $\varepsilon > 0$ one has,*

$$\varepsilon\text{-}\operatorname{argmin}(\varphi - \psi) =$$

$$\bigcap_{\eta>\varepsilon} \bigcup_{r>0} r\text{-}\operatorname{argmin}(\varphi- <,y>) \setminus (r + \delta(y) - \eta)\text{-}\operatorname{argmin}(\psi- <,y>) ,$$

where $\delta(y) = \psi^*(y) - \varphi^*(y) - \inf_Z(\varphi-\psi)$.

When $\psi \in \Gamma_0(Z), \delta(y)$ may be chosen arbitrarily small for:

$$\inf\{\delta(y) : y \in \operatorname{dom} \varphi^* \cap \operatorname{dom} \psi^*\} = 0 \qquad ([23]\ [24]) .$$

Moreover:

COROLLARY 1.18. *If there exists $\overline{y} \in \operatorname{dom} \varphi^* \cap \operatorname{dom} \psi^*$ such that $\inf_Z(\varphi-\psi) = \psi^*(\overline{y}) - \varphi^*(\overline{y})$, then, for all $\varepsilon \geq 0$,*

$$\varepsilon\text{-}\operatorname{argmin}(\varphi-\psi) = \bigcap_{\eta>\varepsilon} \bigcup_{r>0} r\text{-}\operatorname{argmin}(\varphi- <,\overline{y}>) \setminus (r-\eta)\text{-}\operatorname{argmin}(\psi- <,\overline{y}>) .$$

Let us now consider the case of an affine composite function. Here (X,W) and (Z,Y) are two couples of paired ℓ.c.s., $L : X \to Z$ a continuous linear application, $L^* : Y \to W$ its transpose, $z_0 \in Z$, $A(x) = z_0 + L(x)$ for any $x \in X$, $\varphi : Z \to \mathcal{R} \cup \{+\infty\}$. We assume that the affine composite function,

$$x \in X \longmapsto (\varphi \circ A)(x) = \varphi(z_0 + Lx) \in \mathcal{R} \cup \{+\infty\} ,$$

is proper and minorized. To each $y \in \ker L^* \cap (\varphi^*)^{-1}(\mathcal{R})$ we associate the nonnegative real number

$$\delta(y) = \varphi^*(y) - < z_0, y > + \inf_X \varphi \circ A .$$

We then have, for any $\varepsilon \geq 0$ and any $y \in \ker L^* \cap (\varphi^*)^{-1}(\mathcal{R})$,

$$\varepsilon\text{-argmin}\, \varphi \circ A = \{x \in X : \varphi(z_0 + Lx) - < z_0 + Lx, y > + \varphi^*(y) \leq \varepsilon + \delta(y)\}$$

As the function $\varphi - <, y > + \varphi^*(y)$ admits zero for infimum, we have,

$$\varepsilon\text{-argmin}\, \varphi \circ A = \{x \in X : z_0 + Lx \in (\varepsilon + \delta(y))\text{-argmin}(\varphi - <, y >\} .$$

Therefore, we have proved:

PROPOSITION 1.19. *Let $A : X \to Z$ be an affine continuous mapping defined by $A(x) = L(x) + z_0$ for any $x \in X$, with $L : X \to Z$ linear, and $\varphi : Z \to \mathcal{R} \cup \{+\infty\}$ a function. Assume that $\varphi \circ A$ is proper and minorized. Then, for any $y \in \ker L^* \cap (\varphi^*)^{-1}(\mathcal{R})$ we have, for all $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin}\, \varphi \circ A = A^{-1}((\varepsilon + \delta(y))\text{-argmin}(\varphi - <, y >)) ,$$

where $\delta(y) = \varphi^*(y) - < z_0, y > + \inf_X \varphi \circ A$.

When φ is convex and φ is finite and continuous at a point of $A(X)$ there exists $\overline{y} \in \ker L^* \cap \operatorname{dom} \varphi^*$ such that ([22]) $\delta(\overline{y}) = 0$. In such a case we obtain:

COROLLARY 1.20. *Let A, L, φ be as in Proposition 19. Assume, moreover, that φ is convex, finite and continuous at a point of $A(X)$. Then, there exists $\overline{y} \in \ker L^* \cap \operatorname{dom} \varphi^*$ such that, for all $\varepsilon \geq 0$,*

$$\varepsilon\text{-argmin}\, \varphi \circ A = A^{-1}(\varepsilon\text{-argmin}(\varphi - <, \overline{y} >)) .$$

2. Approximate Subdifferential Calculus

2b. COMPLEMENTS ON THE APPROXIMATE SUBDIFFERENTIALS OF AN INF-CONVOLUTION AND AN IMAGE FUNCTION

In this section X and Z are two ℓ.c.s. paired in duality with W and Y respectively, $L : X \to Z$ a linear continuous operator, $L^* : Y \to W$ its transpose. We will give new results about the approximate subdifferentials of the inf-convolution of the proper functions $f, g : X \to \mathcal{R} \cup \{+\infty\}$:

$$(f \Box g)(x) = \inf_{u \in X} (f(x - u) + g(u)) \qquad \text{for all} \quad x \in X . \tag{8}$$

Also, we consider the image of f under L (e.g. [21] p. 38), namely:

$$f_L(z) = \inf\{f(u) : Lu = z\} \quad \text{for all} \quad z \in Z \ . \tag{9}$$

When the infima are attained in (8) and (9), that is when there exist $u_1, u_2 \in X$ such that,

$$(f \,\Box\, g)(x) = f(x - u_1) + g(u_1) \in \mathcal{R} \ ,$$

$$f_L(z) = f(u_2) \in \mathcal{R} \quad \text{with} \quad Lu_2 = z \ ,$$

then, the approximate subdifferentials can be expressed as follows ([9]),

$$\partial_\varepsilon (f \,\Box\, g)(x) = \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial_{\varepsilon_1} f(x - u_1) \cap \partial_{\varepsilon_2} g(u_1) \ , \tag{10}$$

$$\partial_\varepsilon f_L(z) = (L^*)^{-1}(\partial_\varepsilon f(u_2)) \ . \tag{11}$$

In such a case, we observe that the knoweldge of the approximate subdifferentials of f and g (resp. f) at a well chosen point is sufficient for computing $\partial_\varepsilon (f \,\Box\, g)(x)$ (resp. $\partial_\varepsilon f_L(x)$).

Recently, the case when the infima in (8) and (9) are not attained has been investigated in [12], [19]. It appears that the knowledge of f and g (resp. g) at many points is required: namely,

$$\partial (f \,\Box\, g)(x) = \bigcap_{\varepsilon > 0} \ \bigcup_{u \in X} \partial_\varepsilon f(x - u) \ \cap \ \partial_\varepsilon \, g(u) \qquad \text{([12] Thm. 1.1)} \ ;$$

$$\partial_\varepsilon f_L(z) = \bigcap_{\delta > 0} \ \bigcup_{Lu = z} (L^*)^{-1}(\partial_{\varepsilon+\delta} f(u)) = \bigcap_{\delta > 0} \ \bigcap_{u \in S(\delta)} (L^*)^{-1}(\partial_{\varepsilon+\delta} f(u))$$

([18], Thm. 1),

with $S(\delta) := \{u \in X : Lu = z, \ f(u) \leq f_L(z) + \delta\}$. The next propositions show that, in spite of the fact that the infima are not necessarily attained in (8) and (9), formulas similar to (10) and (11) hold (for the convex case see also [14] Thm 4.2.8).

PROPOSITION 2.1. *Let $f, g : X \to \mathcal{R} \cup \{+\infty\}$ be proper functions and x a point of X where $(f \,\Box\, g)\,(x)$ is finite. Then, for any $u \in X$ such that $f(x-u)+g(u) \in \mathcal{R}$, we have, for all $\varepsilon \geq 0$,*

$$\partial_\varepsilon (f \,\Box\, g)(x) = \bigcup_{\substack{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \delta}} \partial_{\varepsilon_1} f(x - u) \cap \partial_{\varepsilon_2} g(u) \ ,$$

where $\delta := f(x - u) + g(u) - (f \,\Box\, g)(x)$.

Proof. Taking into account the classical formula $(f \square g)^* = f^* + g^*$, we have $w \in \partial_\varepsilon (f \square g)(x)$ iff (see (1))

$$f^*(w) + g^*(w) - < x, w > + (f \square g)(x) \leq \varepsilon ,$$

that is

$$[f^*(w) - < x - u, w > + f(x - u)] + [g^*(w) - < u, w > + g(u)] \leq \varepsilon + \delta .$$

By Fenchel inequality the numbers into the brackets are nonnegative. Hence, by (3), the above line is equivalent to the existence of $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, $\varepsilon_1 + \varepsilon_2 = \varepsilon + \delta$, such that,

$$f^*(w) - < x - u, w > + f(x - u) \leq \varepsilon_1, \ g^*(w) - < u, w > + g(u) \leq \varepsilon_2 ,$$

or, in other words, $w \in \partial_{\varepsilon_1} f(x - u) \cap \partial_{\varepsilon_2}(u)$. ∎

In the same way one has (see also the proof of Cor. 2 in [19]):

PROPOSITION 2.2. *Let f, L be as above, and $z \in (f_L)^{-1}(\mathcal{R})$. Then, for any $u \in X$ such that $Lu = z$ and $f(u) \in \mathcal{R}$, we have, for all $\varepsilon \geq 0$,*

$$\partial_\varepsilon f_L(z) = (L^*)^{-1}(\partial_{\varepsilon+\delta} f(u)) ,$$

where $\delta := f(u) - f_L(z)$.

Proof. From the well known formula $(f_L)^* = f^* \circ L^*$, it follows that $y \in \partial_\varepsilon f_L(z)$ iff,

$$f^*(L^* y) - < z, y > + f_L(z) \leq \varepsilon ,$$

that is,

$$f^*(L^* y) - < Lu, y > + f(u) = f^*(L^* y) - < u, L^* y > + f(u) \leq \varepsilon + \delta .$$

This says exactly that $L^* y \in \partial_{\varepsilon+\delta} f(u)$, i.e. $y \in (L^*)^{-1}(\partial_{\varepsilon+\delta} f(u))$. ∎

2.2. SUBDIFFERENTIAL OF AN UPPER ENVELOPE OF CONVEX FUNCTIONS

Let us consider an arbitrary family $(f_\alpha)_{\alpha \in A}$ of proper convex functions defined on a topological vector space X. There exists a formula, due to Valadier [25], for the subdifferential of the upper envelope,

$$f = \sup_{\alpha \in A} f_\alpha ,$$

at any point $x \in X$ where f is finite and continuous. In the case when X is normed this formula may be written,

$$\partial f(x) = \bigcap_{\eta > 0} \overline{\mathrm{co}} \, \{\partial f_\alpha(u) : \alpha \in A_\eta \ , \ u \in x + \eta B\} ,$$

where,

$$A_\eta = \{\alpha \in A : f_\alpha(x) \geq f(x) - \eta\} ,$$

and B is the unit ball of X.

It has been shown in [29] that $\partial f(x)$ can be expressed in terms of the approximate subdifferentials of the f_α at the same point x. The proof in [29] is based on Valadier formula. Here we propose a direct proof which uses our previous results about ε-argmin calculus, but we restrict ourselves to the finite dimensional case $X = \mathcal{R}^n$. We retain the same hypothesis, that is,

$$f \text{ is finite and continuous at } x \in \mathcal{R}^n .$$

It follows that there exists an open neighbourhood V of x such that each f_α is majorized on V and, consequently,

$$f_\alpha(u) = f_\alpha^{**}(u) \ \forall \alpha \in A, \ \ \forall u \in V .$$

Due to the local character of the exact subdifferential, we then have,

$$\partial f(x) = \partial(\sup_{\alpha \in A} f_\alpha^{**})(x) .$$

Now let us set,

$$\varphi = \inf_{\alpha \in A} f_\alpha^* .$$

One has,

$$\varphi^* = \sup_{\alpha \in A} f_\alpha^{**} ,$$

and, by Proposition 0.2,

$$\begin{aligned} \partial f(x) &= \text{argmin } (\varphi^{**} - < x, . >) \\ &= \text{argmin } (\varphi - < x, . >)^{**} \\ &= \text{argmin } (\inf_{\alpha \in A} \psi_\alpha)^{**} , \end{aligned} \tag{12}$$

where,

$$\psi_\alpha := f_\alpha^* - < x, . > ,$$

so that, for any $\eta \geq 0$,

$$\eta\text{-argmin } \psi_\alpha = \partial_\eta f_\alpha^{**}(x) \ = \ \partial_\eta f_\alpha(x) . \tag{13}$$

Let us introduce the function,

$$\psi = \inf_{\alpha \in A} \psi_\alpha .$$

As ψ and its ℓ.s.c. hull $\overline{\psi}$ have the same closed convex hull, one has by (12),

$$\partial f(x) = \text{argmin}(\overline{\psi})^{**} .$$

Now, the continuity of the convex function f at $x \in \text{dom}\, f$ amounts to the existence of $\varepsilon > 0$ and $c \in \mathcal{R}$ such that,

$$f \leq I_{x+\varepsilon B} - c , \tag{14}$$

where, for any $u \in \mathcal{R}^n$, $I_{x+\varepsilon B}(u) = 0$ if $x - u \in \varepsilon B$, $+\infty$ if not.

By taking the Fenchel transforms in both member of (14) we get,

$$\overline{\psi} \geq f^* - < x, . > \geq \varepsilon \| \, \| + c ,$$

so that $\overline{\psi}$ does satisfy the condition (5) required in Proposition 1.14.

Hence we have,

$$\partial f(x) = \text{co argmin}\, \overline{\psi} ,$$

and, by Proposition 1.13,

$$\partial f(x) = \text{co} \bigcap_{\varepsilon>0} \overline{\varepsilon\text{-argmin}\, \psi} \subset \bigcap_{\varepsilon>0} \overline{\text{co}}(\varepsilon\text{-argmin}\, \psi) .$$

Moreover, by lemma 1.7,

$$\varepsilon\text{-argmin}\, \psi \subset \bigcap_{\eta>\varepsilon} \bigcup_{\alpha\in A_\eta} \eta\text{-argmin}\, \psi_\alpha .$$

From (13) we obtain,

$$\partial f(x) \subset \bigcap_{\varepsilon>0} \overline{\text{co}} \bigcap_{\eta>\varepsilon} \bigcup_{\alpha\in A_\eta} \partial_\eta f_\alpha(x) .$$

Finally, we get the more simpler inclusion,

$$\partial f(x) \subset \bigcap_{\eta>0} \overline{\text{co}} \bigcup_{\alpha\in A_\eta} \partial_\eta f_\alpha(x) .$$

A very nice thing is that the reverse inclusion holds ! As a matter of fact we do have for any $\eta > 0$, $\alpha \in A_\eta$,

$$\partial_\eta f_\alpha(x) \subset \partial_{2\eta} f(x) .$$

Indeed, $w \in \partial_\eta f_\alpha(x)$ entails, for all $u \in \mathcal{R}^n$,

$$f(u) \geq f_\alpha(u) \geq f_\alpha(x) + < u - x, w > -\eta \geq f(x) + < u - x, w > -2\eta .$$

As $\partial_{2\eta} f(x)$ is closed convex, it ensues that,

$$\overline{\text{co}} \bigcup_{\alpha \in A_\eta} \partial_\eta f_\alpha(x) \subset \partial_{2\eta} f(x) ,$$

and the announced inclusion follows by taking the intersection over all $\eta > 0$.

Hence we have proved:

THEOREM 2.3. *Let $(f_\alpha)_{\alpha \in A}$ be a family of proper convex functions on $\mathcal{R}^n$. Assume that,*

$$f = \sup_{\alpha \in A} f_\alpha$$

is finite and continuous at a given point x of $\mathcal{R}^n$. Then,

$$\partial f(x) = \bigcap_{\eta > 0} \overline{\text{co}} \{\partial_\eta f_\alpha(x) : \alpha \in A_\eta\} , \tag{15}$$

with $A_\eta := \{\alpha \in A : f_\alpha(x) \geq f(x) - \eta\}$.

The formula (15) can be simplified in the following situations:

THEOREM 2.4. *Let $(f_\alpha)_{\alpha \in A}$, f, and x be as in Theorem 2.3. Assume, moreover, that the function,*

$$y \in \mathcal{R}^n \longmapsto \inf_{\alpha \in A} f_\alpha^*(y) \quad \text{is } \ell\text{.s.c.}.$$

Then,

$$\partial f(x) = \bigcap_{\eta > 0} \text{co} \{\partial_\eta f_\alpha(x) : \alpha \in A_\eta\} .$$

If, in addition,

$$\forall y \in \partial f(x) , \quad \exists \beta \in A : \inf_{\alpha \in A} f_\alpha^*(y) = f_\beta^*(y) ,$$

then,

$$\partial f(x) = \text{co} \bigcup_{\alpha \in A_0} \partial f_\alpha(x) ,$$

where, for any $\eta \geq 0$, $A_\eta = \{\alpha \in A : f_\alpha(x) \geq f(x) - \eta\}$.

Proof. By assumption, the function $\psi = \inf_{\alpha \in A} \psi_\alpha$, $\psi_\alpha = f_\alpha^* - < x, . >$, is ℓ.s.c. Following the same way as in Theorem 2.3 we obtain,

$$\partial f(x) = \text{coargmin}\, \psi \subset \bigcap_{\eta > 0} \text{co} \bigcup_{\alpha \in A_\eta} \eta\text{-argmin}\, \psi_\alpha ,$$

hence,

$$\partial f(x) \subset \bigcap_{\eta>0} \text{co} \bigcup_{\alpha\in A_\eta} \partial_\eta f_\alpha(x) \subset \bigcap_{\eta>0} \partial_{2\eta} f(x) = \partial f(x) .$$

To prove the second formula, we use the second part of Lemma 1.8, noticing that argmin $\psi \subset \partial f(x)$:

$$\begin{aligned} \partial f(x) &= \text{ coargmin } \psi = \text{ co } \bigcup_{\alpha\in A_0} \text{ argmin } \psi_\alpha \\ &= \text{ co } \bigcup_{\alpha\in A_0} \partial f_\alpha(x) . \end{aligned}$$

■

REMARK. A standard way to satisfy the assumptions of the above Theorem is to suppose that A is a topological compact space, and for each $z \in \mathcal{R}^n$, the function $\alpha \in A \longmapsto f_\alpha(z)$ is upper-semicontinuous. Such a condition is used in [13] Theorem 4.4.2 to obtain the second formula in Theorem 2.4. Here, our proof is totally different.

When dealing with the upper envelope of just two ℓ.s.c. convex functions from $\mathcal{R}^n$ to $\mathcal{R} \cup \{+\infty\}$,

$$f = \max(f_1, f_2) = f_1 \vee f_2 ,$$

another formula may be derived from the argmin formula (7).

Let us assume that f is finite and continuous at some point $\overline{x}$ of $\mathcal{R}^n$. In other words,

$$\exists \varepsilon > 0, \quad \exists c \in \mathcal{R} : \ f \leq I_{\overline{x}+\varepsilon B} - c ,$$

or, equivalently, with $f_1^* \wedge f_2^* = \min(f_1^*, f_2^*)$,

$$f_1^* \wedge f_2^* - < \overline{x}, . > \geq f^* - < \overline{x}, . > \geq \varepsilon \| \ \| + c .$$

This means that $f_1^* \wedge f_2^*$ is epi-pointed in the sense of [4].

Let us now consider $x \in \text{dom } f_1 \cap \text{ dom } f_2$ such that,

$$f_1(x) = f_2(x) ,$$

and set,

$$\psi_i = f_i^* - < x, . > \qquad (i = 1, 2) .$$

By (7) we then obtain,

$$\text{argmin}(\psi_1 \wedge \psi_2)^{**} = \text{ co argmin}(\psi_1 \wedge \psi_2) + \text{ co argmin}(\psi_1 \wedge \psi_2)_\infty . \quad (16)$$

Now we have, by introducing the asymptotic cone,

$$A_\infty = \bigcap_{\varepsilon>0} \overline{[0,\varepsilon]A} \,,$$

of a subset A of a topological space ([5], [6], ...), and denoting by $E(\varphi)$ the epigraph of a functional φ,

$$\begin{aligned} E(\psi_1 \wedge \psi_2)_\infty &= (E(\psi_1 \wedge \psi_2))_\infty && \text{(see, e.g. [6])} \\ &= (E(\psi_1) \cup E(\psi_2))_\infty \\ &= E(\psi_1)_\infty \cup E(\psi_2)_\infty && \text{([8] Prop. 1.2)} \,. \end{aligned}$$

Consequently,

$$(\psi_1 \wedge \psi_2)_\infty = (\psi_1)_\infty \wedge (\psi_2)_\infty \qquad \text{(see also [4])} \,. \tag{17}$$

On the other hand, the functions $\psi_1, \psi_2, (\psi_1)_\infty, (\psi_2)_\infty$ admit zero for infimum. By (16) and (17) we then obtain,

$$\operatorname{argmin}(\psi_1 \wedge \psi_2)^{**} =$$

$$\operatorname{co}(\operatorname{argmin} \psi_1 \cup \operatorname{argmin} \psi_2) + \operatorname{co}(\operatorname{argmin}(\psi_1)_\infty \cup \operatorname{argmin}(\psi_2)_\infty) \,.$$

Now, for $i = 1, 2$,

$$\operatorname{argmin} \psi_i = \partial f_i(x) \qquad \text{(see Prop. 0.2))} \,,$$

while $(\psi_i)_\infty$ is the support function (e.g. [15]) of

$$\operatorname{dom} \psi_i^* = -x + \operatorname{dom} f_i \,.$$

From this fact, we get,

$$\operatorname{argmin}(\psi_i)_\infty = \{w \in \mathcal{R}^n : \forall u \in \operatorname{dom} f_i :< u - x, w > \leq 0\} \,,$$

the normal cone of f_i at x (e.g. [15]):

$$\operatorname{argmin}(\psi_i)_\infty = N(\operatorname{dom} f_i, x) \,.$$

As,

$$\operatorname{co}(N(\operatorname{dom} f_1, x) \cup N(\operatorname{dom} f_2, x)) = N(\operatorname{dom} f_1, x) + N(\operatorname{dom} f_2, x) \,,$$

we can state:

THEOREM 2.5. *For any ℓ.s.c. convex functions f_1, f_2 from $\mathcal{R}^n$ to $\mathcal{R} \cup \{+\infty\}$,* *finite and continuous at a same point, we have, at each point x of $\mathcal{R}^n$ such that $f_1(x) = f_2(x)$,*

$$\partial(f_1 \vee f_2)(x) = \operatorname{co}(\partial f_1(x) \cup \partial f_2(x)) + N(\text{dom } f_1, x) + N(\text{dom } f_2, x) .$$

REMARKS. 1) If f_1 and f_2 are finite and continuous at x, then $N(\text{dom } f_1, x) = N(\text{dom } f_2, x) = \{0\}$, and we recover a classical formula ([7]).

2) For a possible extension and improvement of Theorem 2.5 to infinite dimensional spaces see [29] Th. 2, [27], Th. 6 bis.

2.3. SUBDIFFERENTIAL OF A DECONVOLUTION

Of course, Theorem 2.3 can be applied to compute the subdifferential of the deconvolution (see Section 1) of two proper functions f, g on $\mathcal{R}^n$. In such a case, we have to take $A = \text{dom } g$ and, for all $u \in A$,

$$f_u(.) = f(. + u) - g(u) .$$

Assuming f convex and $f \boxminus g$ finite and continuous at x we then obtain,

$$\partial(f \boxminus g)(x) = \bigcap_{\eta>0} \overline{\operatorname{co}} \bigcup_{u \in A_\eta} \partial_\eta f(x+u) ,$$

with $A_\eta = \{u \in \mathcal{R}^n : f(x+u) - g(u) \geq (f \boxminus g)(x) - \eta\}$.

Moreover, noticing that, for any $y \in \mathcal{R}^n$,

$$\begin{aligned} \inf_{u \in \text{ dom } g} (f_u)^*(y) &= f^*(y) + \inf_{u \in \text{ dom } g} (g(u) - \langle u, y \rangle) \\ &= f^*(y) - g^*(y) , \end{aligned}$$

we can applied the second formula of Theorem 2.4 provided $f^* - g^*$ is assumed to be ℓ.s.c. and,

$$\partial f(x) \subset \bigcup_{u \in \text{ dom } g} \partial g(u) := \partial g(\mathcal{R}^n) .$$

In such a case we then have,

$$\partial(f \boxminus g)(x) = \operatorname{co} \bigcup_{u \in A_0} \partial f(x+u) ,$$

with $A_0 = \{u \in \text{dom } g : f(x+u) - g(u) = (f \boxminus g)(x)\}$.

We have therefore proved:

THEOREM 2.6. *Let f and g be proper functions on $\mathcal{R}^n$, with f convex, $f^* - g^*$ ℓ.s.c., and, dom $f^* \subset \partial g(\mathcal{R}^n)$. Then, at each point x of $\mathcal{R}^n$ where $f \boxminus g$ is finite and continuous,*

$$\partial(f \boxminus g)(x) = \operatorname{co} \bigcup_{u \in A_0} \partial f(x+u) .$$

REMARKS. (1) The assumptions of Theorem 2.6 are universally satisfied when g is ℓ.s.c. and strongly coercive,

$$\lim_{||u||\to+\infty} \frac{g(u)}{||u||} = +\infty ,$$

for we then have $\partial g(\mathcal{R}^n) = \mathcal{R}^n$.

(2) In Theorem 2.6, the functions,

$$u \in \mathcal{R}^n \longmapsto f(u+z) - g(u), \quad z \in \mathcal{R}^n ,$$

are not necessarily u.s.c., so that the corresponding assumption of [13] Thm 4.4.2 is not satisfied.

Due to the specific form of the deconvolution, another approach is possible. For this, we shall assume that f and g belong to $\Gamma_0(X)$, where X is a ℓ.c.s. paired in duality with another ℓ.c.s. W . It has been shown in [28] that, in the special case when g^* is continuous and finitely valued and $f^* - g^*$ is convex, $\partial f \boxminus g$ appears to be the parallel star difference of ∂f and ∂g. Namely:

$$w \in \partial(f \boxminus g)(x) \Longleftrightarrow x \in \partial f^*(w) \overset{*}{-} \partial g^*(w) .$$

More explicitely,

$$\begin{aligned}\partial(f \boxminus g)(x) &= \{w \in W : x + \partial g^*(w) \subset \partial f^*(w)\} \\ &= \{w \in W : \forall u \in X : w \in \partial g(u) \Longrightarrow w \in \partial f(x+u)\} ,\end{aligned}$$

so that w does not belong to $\partial(f \boxminus g)(x)$ iff there exists $u \in X$ such that $w \in \partial g(u) \cap ((W \setminus \partial f(x+u))$.

Hence we have proved:

PROPOSITION 2.7. *Let X, W be ℓ.s.c. in duality, f, $g \in \Gamma_0(X)$. Assume that g^* is finite over W and that $f^* - g^* \in \Gamma_0(W)$. Then, for all $x \in X$,*

$$\partial(f \boxminus g)(x) = \bigcap_{u \in X} \partial f(x+u) \cup (W \setminus \partial g(u)) .$$

Concerning the approximate subdifferentials one has:

PROPOSITION 2.8. *Let $f, g \in \Gamma_0(X)$, and assume that $f^* - g^* \in \Gamma_0(W)$. Then, for any $\varepsilon > 0$,*

$$\partial_\varepsilon(f \boxminus g)(x) = \bigcap_{\lambda>0} \bigcap_{u \in X} \partial_{\varepsilon+\lambda} f(x+u) \cup (W \setminus \partial_\lambda g(u)) . \tag{18}$$

Proof. As $f^* - g^* \in \Gamma_0(W)$, it follows that ([10]),

$$(f \boxminus g)^* = f^* - g^* .$$

Hence, for each $\varepsilon \geq 0$,

$$w \in \partial_\varepsilon(f \boxminus g)(x) \Longleftrightarrow x \in \partial_\varepsilon(f^* - g^*)(w) .$$

Applying ([16], Thm. 1), we get,

$$\partial_\varepsilon(f^* - g^*)(w) = \bigcap_{\lambda>0} \partial_{\varepsilon+\lambda} f^*(w) \overset{*}{-} \partial_\lambda g^*(w)$$

so that, following the same lines as for $\varepsilon = 0$, we obtain (18).

When the stringent assumption $f^* - g^*$ convex is not satisfied, one must take another way. We still assume that f and g belong to $\Gamma_0(\mathcal{R}^n)$ and, in addition, that

$$h = f \boxminus g \quad \text{is finite and continuous at } x \in \mathcal{R}^n . \tag{19}$$

Now,

$$\begin{aligned} \partial h(x) &= \operatorname{argmin} h^* - < x, . > && (\text{as } h \in \Gamma_0(\mathcal{R}^n)) \\ &= \operatorname{argmin}(f^* - g^* - < x, . >)^{**} && (\text{as } h^* = (f^* - g^*)^{**}) \\ &= \operatorname{co}(\operatorname{argmin}(f^* - g^* - < x, . >)) && (\text{by (19) and Proposition 1.14}) \end{aligned}$$

For any $u \in \operatorname{dom} g$ (whence $x + u \in \operatorname{dom} f$), let us introduce the functions,

$$\varphi = f^* - < x + u, . > , \qquad \psi = g^* - < u, . > .$$

We have $\inf_{\mathcal{R}^n} \varphi = -f(x+u)$, $\inf_{\mathcal{R}^n} \psi = -g(u)$, r-argmin$\varphi = \partial_r f(x+u)$ s-argmin$\psi = \partial_s g(u)$, so that, by Proposition 1.2,

$$\operatorname{argmin}(\varphi - \psi) = \operatorname{co} \bigcap_{\eta>0} \bigcup_{r>0} \partial_r f(x+u) \setminus \partial_s g(u) ,$$

with,

$$s := r - \eta + (f \boxminus g)(x) - f(x+u) + g(u) . \tag{20}$$

It follows that,

$$\partial h(x) \subset \bigcap_{\eta>0} \operatorname{co} \bigcup_{r>0} \partial_r f(x+u) \setminus \partial_s g(u) . \tag{21}$$

To prove the other inclusion in (21), let us take $\delta > 0$, $r > 0$, and $u \in \operatorname{dom} g$; for any $w \in \partial_r f(x+u) \setminus \partial_s g(u)$ there exists $v \in \mathcal{R}^n$ such that for all $z \in \mathcal{R}^n$:

$$f(v+z) - f(x+u) \geq < v + z - x - u, w > -r$$

$$g(u) - g(v) >< u - v, w > +s \ .$$

Taking into account the value of s given by (20), we deduce,

$$h(z) \geq f(v+z) - g(v) \geq < z - u, w > +h(x) - \eta \ ,$$

hence $w \in \partial_\eta h(x)$. Consequently, for any $\eta > 0$, we have,

$$\bigcup_{r>0} \partial_r f(x+u) \setminus \partial_s g(u) \subset \partial_\eta h(x) \ ,$$

and, as $\partial_\eta h(x)$ is convex,

$$\mathrm{co} \bigcup_{r>0} \partial_r f(x+u) \setminus \partial_s g(u) \subset \partial_\eta h(x) \ .$$

Taking the intersection over all $\eta > 0$, we get the opposite inclusion to (21). Hence we have proved:

THEOREM 2.9. *Let f, g be proper convex ℓ.s.c. functions on $\mathcal{R}^n$. Assume that the deconvolution $f \boxminus g$ is finite and continuous at $x \in \mathcal{R}^n$. Then, for any $u \in$ dom g,*

$$\partial(f \boxminus g)(x) = \bigcap_{\eta>0} \mathrm{co} \bigcup_{r>0} \partial_r f(x+u) \setminus \partial_s g(u) \ ,$$

with $s := r - \eta + (f \boxminus g)(x) - f(x+u) + g(u)$.

REMARK. If the deconvolution is exact at x, that is if there exists $u \in$ dom g such that $f(x+u) - g(u) = (f \boxminus g)(x)$, it comes,

$$\partial(f \boxminus g)(x) = \bigcap_{\eta>0} \mathrm{co} \bigcup_{r>0} \partial_r f(x+u) \setminus \partial_{r-\eta} g(u) \ .$$

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